INTRODUCTION

With increasing emphasis on higher reliability, longer warranty periods, minimizing the cost of manufacturing, and liability litigation cases are on the rise, there is a pressing need for refined methods to analyze these durability and compliance problems. Methods are required to determine the seriousness of problems in testing, designing, and field surveillance and being able to predict and drawn meaningful inference from available data.

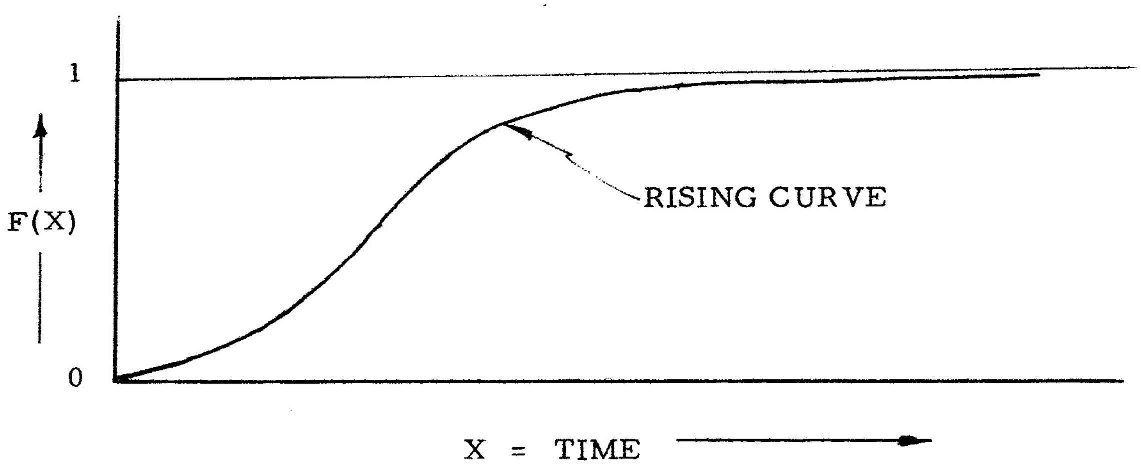
The CDF technique appears to offer considerable promise of answering this need. The CDF stands for Cumulative Distribution Functions which consists of the Weibull, Ultimate Value, and N-tic Power function types. The Weibull statistical technique has gained wide acceptance in the field of life testings since the postulation of a general statistical cumulative function by W. Weibull of Sweden in 1939, and extended by Leonard G. Johnson, formerly of GM Research Laboratories and now of Detroit Research Institute. Extensive studies on various types of CDFs were being made at the Detroit Research Institue, and have come up with two other useful types, the Ultimate Value and N-tic . These two types, we believed will be equally as important as the Weibull for those concerned with fitting failure data (or other types of order statistics). The nice things about these two CDFs are the simplicity of mathematics and straightforward methods. They are quite simple and easy to work with, much in the same way as the Weibull analysis, --- the ability to represent failure data (or other types of order statistics) as straight lines on special probability papers.

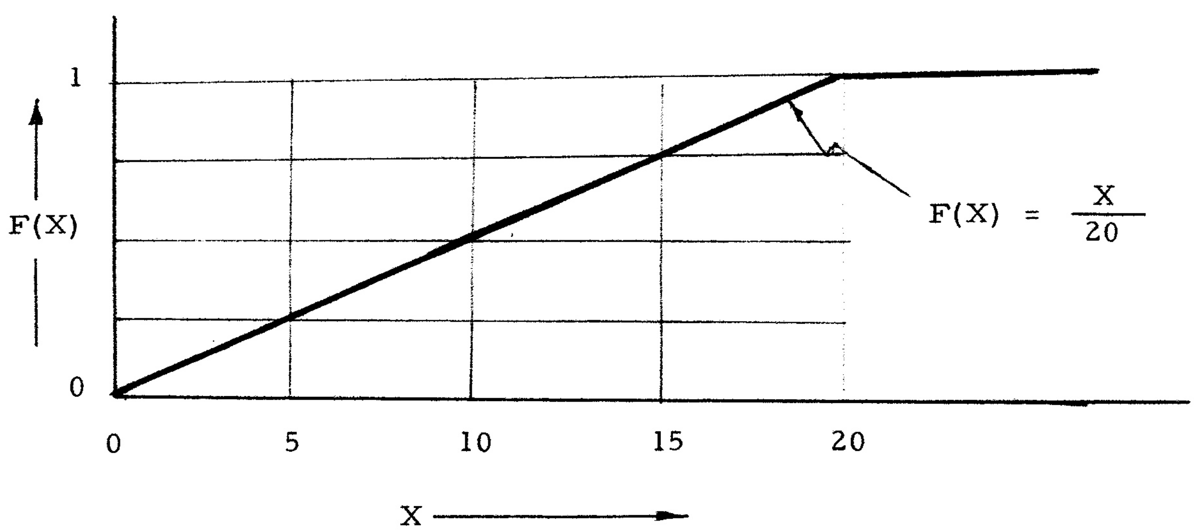
It is the purpose of the illustrated examples which follows to help the user to understand the basis of the program CDF well enough to analyze his problems more effectively and economically.

DEFINITION OF CUMULATIVE DISTRIBUTION FUNCTION

A cumulative distribution function F(X), is a monotone increasing function, where X is positive and must turn out to be some type of rising curve, (never falling back) starting at the zero level and eventually reaching the unity level. ($F(0) = 0$ and $F(\infty) = 1$).

Cumulative distribution function takes on many shapes such as the ones which follows:





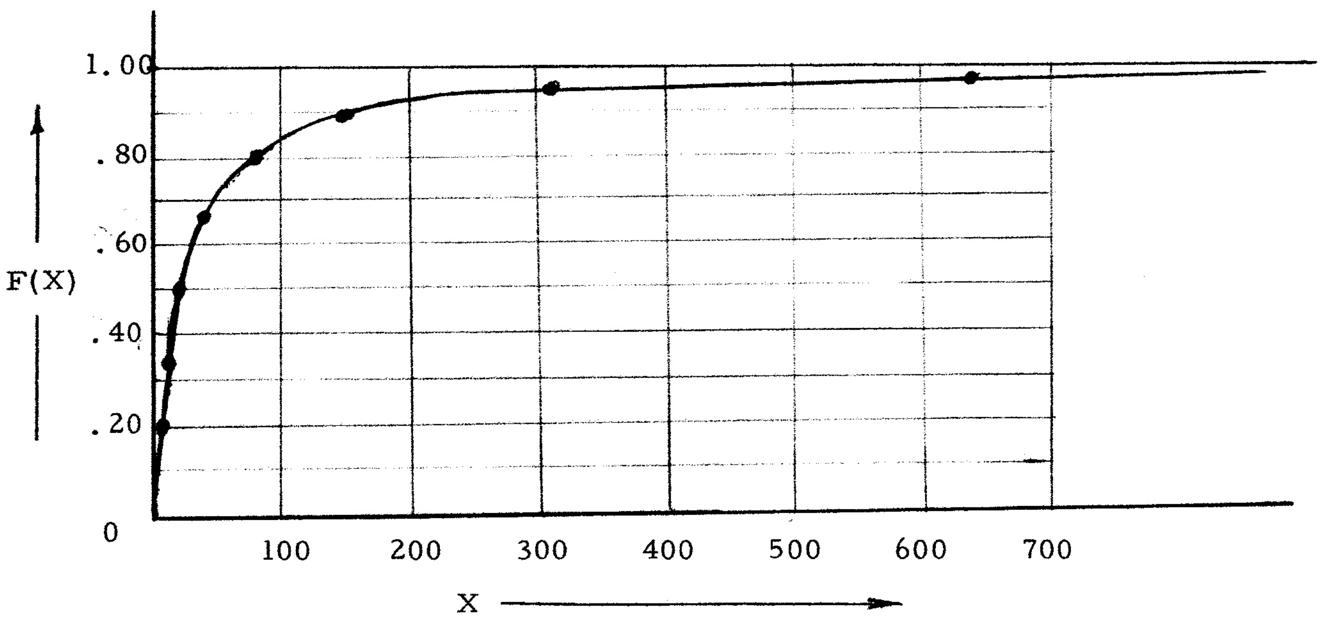
Another example of a cumulative distribution function is

$$F(X) = \frac{X}{X + 20} \quad .$$

To construct the graph of this we form the table

X	F(X)
0	0
5	.200
10	.333
20	.500
40	.667
80	.800
160	.889
320	.941
640	.970

These values are plotted below :



THE CUMULATIVE DISTRIBUTION FUNCTIONS

The CDFs of Weibull , Ultimate Value, and N-tic types are excellent models for describing the life characteristics of a large group of physical and biological problems. This unique feature and the Median Ranks provide us with an expedient method of associating cumulative distributional percentages with observed outcomes as straight lines on special probability papers are the primary reasons for interest in these distributions.

The general expressions for the CDFs are as follows :

1). WEIBULL TYPE :

$$F(X) = 1 - e^{-\left(\frac{X - \alpha}{\theta - \alpha}\right)^{b}}$$

(This is a 3-parameter Weibull)

where

α = the minimum life or location parameter

b = the Weibull Slope or shape parameter

θ = the characteristic life or scale parameter

The minimum life (α) is usually assumed to be zero so the above expression becomes

$$F(X) = 1 - e^{-\left(\frac{X}{\theta}\right)^{b}}$$

(a 2-parameter Weibull)

This 2-parameter Weibull CDF is the basis for constructing straight lines on special Weibull graph paper to represent data. Observed outcomes are plotted against median rank values and a straight line of best fit is drawn to the plotted points.

The procedure for constructing these curves is as follows :

$$F(X) = 1 - e^{-\left(\frac{X}{\Theta}\right)^b}$$

rearranging terms

$$1 - F(X) = e^{-\left(\frac{X}{\Theta}\right)^b}$$

$$\frac{1}{1 - F(X)} = e^{+\left(\frac{X}{\Theta}\right)^b}$$

taking the natural log of both sides , we get

$$\ln \frac{1}{1 - F(X)} = \left(\frac{X}{\Theta}\right)^b$$

taking the natural log of both sides again, the result is :

$$\ln \ln \left(\frac{1}{1 - F(X)}\right) = b \ln X - b \ln \Theta$$

now make the substitution

$$Y = \ln \ln \left(\frac{1}{1 - F(X)}\right)$$

$$X = \ln X$$

$$A = -b \ln \Theta$$

$$B = b$$

then , $Y = BX + A$

which is a linear plot in Y with slope B.

Thus, a Weibull distribution is equivalent to a linear relationship between the logarithm of life accounted for and the logarithm of the logarithm of

$$\frac{1}{1 - \text{Fraction failed within life } X} .$$

2). ULTIMATE VALUE TYPE :

$$F(X) = e^{\left[1 - \left(\frac{U - \alpha}{X - \alpha}\right)^{\gamma}\right]}$$

(3- parameter Ultimate Value)

where

α = the minimum life value or location parameter

γ = the Ultimate Value slope or shape paramter

U = Ultimate Value

Assuming $\alpha = 0$, the function becomes

$$F(X) = e^{\left[1 - \left(\frac{U}{X}\right)^{\gamma}\right]}$$ (a 2-parameter Ultimate Value)

The procedure for constructing the Ultimate Value curves on Ultimate Value graph paper is as follows by using the 2-parameter distribution.

$$F(X) = e^{\left[1 - \left(\frac{U}{X}\right)^{\gamma}\right]}$$

$$\ln F(X) = 1 - \left(\frac{U}{X}\right)^{\gamma}$$

$$1 - \ln F(X) = \left(\frac{U}{X}\right)^{\gamma}$$

$$\frac{1}{1 - \ln F(X)} = \left(\frac{X}{U}\right)^{\gamma}$$

$$\ln\left(\frac{1}{1 - \ln F(X)}\right) = \gamma \ln X - \gamma \ln U$$

let

$$Y = \ln\left(\frac{1}{1 - \ln F(X)}\right)$$

$$X = \ln X$$

$$A = - \gamma \ln U$$

$$B = \gamma$$

then , $Y = BX + A$ (a linear plot)

3). N-TIC TYPE (N^{th} Order Power Function) :

$$F(X) = \left(\frac{X - \alpha}{L - \alpha} \right)^n \qquad n \geqslant 0, \quad \alpha \geqslant 0, \quad L \geqslant 0$$

(3-parameter N-tic)

where

α = minimum life value

n = N-tic slope or degree of the CDF of the N-tic population

L = Maximum life value

The procedure for constructing the N-tic function curves on log-log graph paper is as follows by using the 2-parameter distribution ,

$$F(X) = \left(\frac{X}{L} \right)^n \qquad (\text{assuming } \alpha = 0)$$

$$F(X) = \left(\frac{X}{L} \right)^n$$

$$\ln F(X) = n \ln X - n \ln L$$

make the substitution by letting

$Y = \ln F(x)$

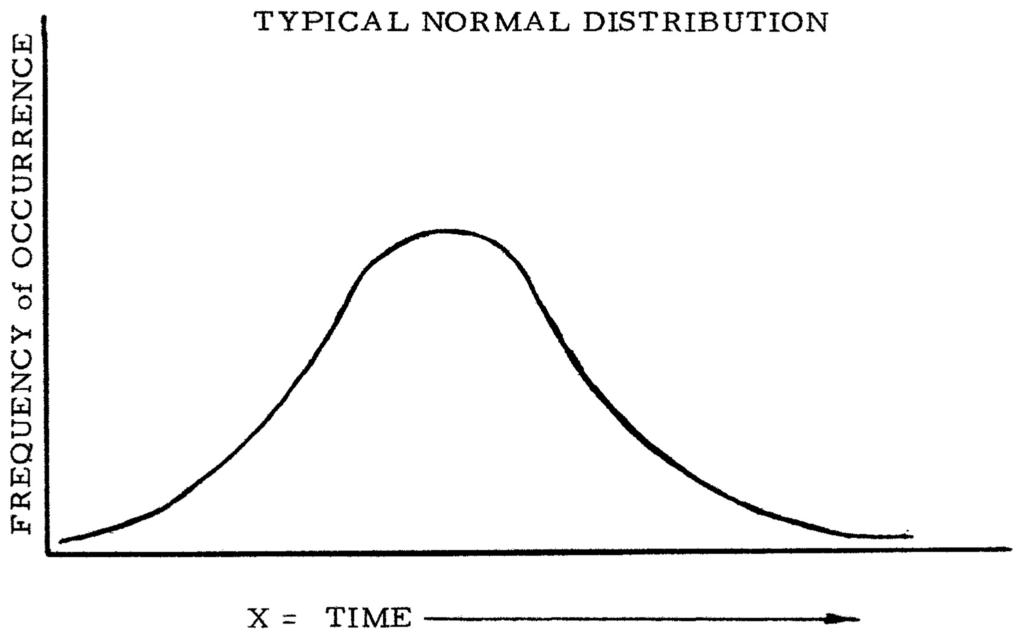
$A = - n \ln L$

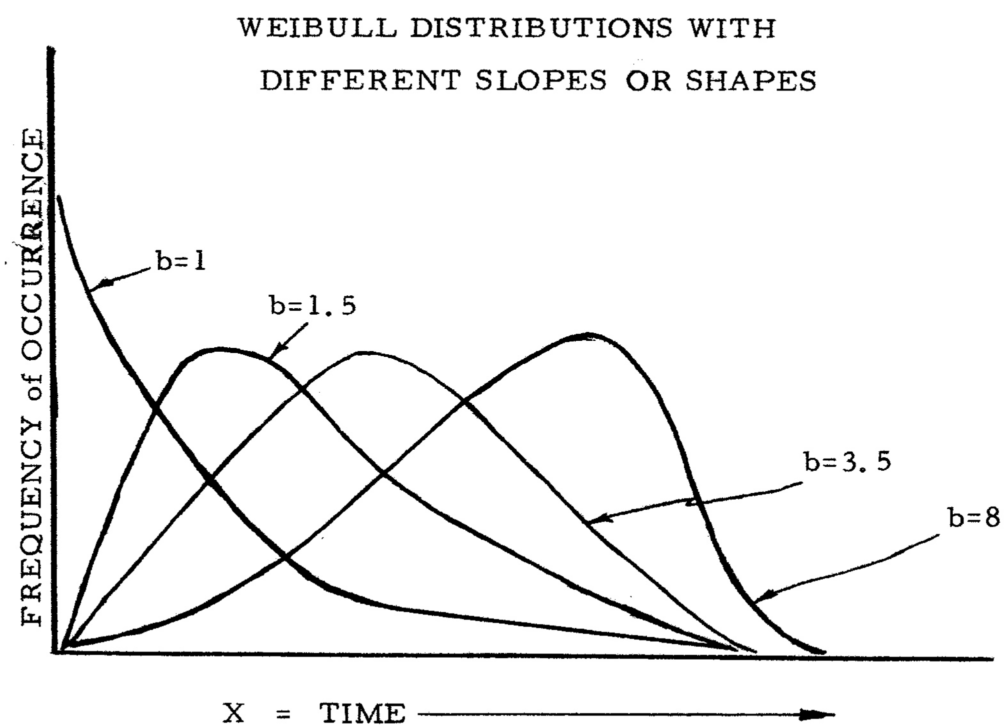
$X = \ln X$

$B = n$

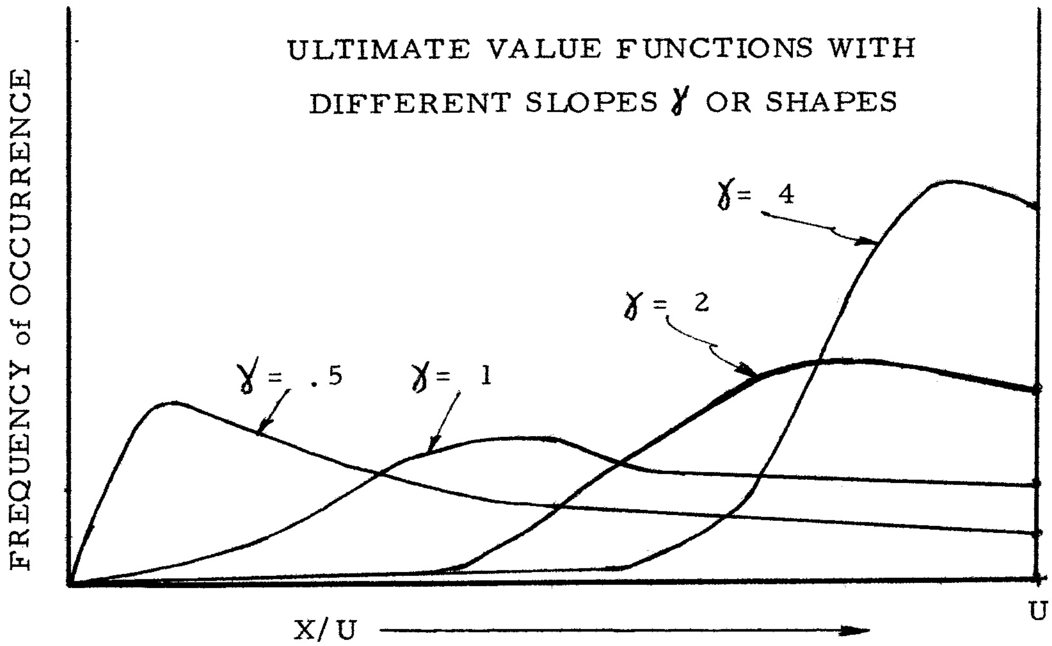
then $Y = BX + A$

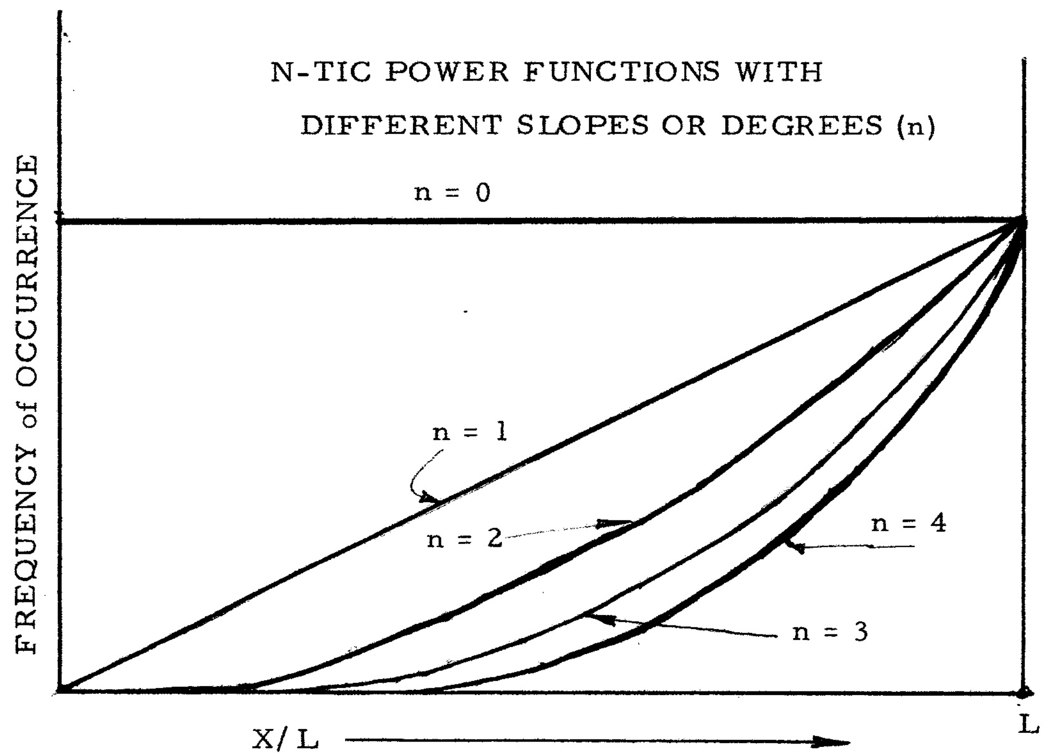
This is again also a linear plot in Y with slope B.


TYPICAL NORMAL DISTRIBUTION
FREQUENCY of OCCURRENCE
X = TIME


WEIBULL DISTRIBUTIONS WITH
DIFFERENT SLOPES OR SHAPES
FREQUENCY of OCCURRENCE
b=1
b=1.5
b=3.5
b=8
X = TIME

PROBABILITY DENSITY FUNCTIONS (PDF) --- FREQUENCY CURVES





AREA OF APPLICATIONS

I WEIBULL DISTRIBUTION

A). Recognized Applications

$b < 1$ (work hardening failures)

$b = 1$ (random failures)

$b > 1$ (wearout failures)

B). Other Applications

Marketing (auction prices ; discount prices ; i. e. , eventually, with enough discount, the customer fails to resist.)

II ULTIMATE VALUE DISTRIBUTION

A). Static Failures (Where there is an ultimate load which always causes failure.)

B). Finance and Economics (Where there is an ultimate level, which once reached, is followed by a decline.)

III N-TIC DISTRIBUTION

A). Business Applications
(Marketing Bid Theory)

B). Reliability Applications
(Very early failures)
In a Weibull Population, the very smallest values have an N-Tic Distribution.